Effect of Decoherence on Bell's Inequality for an EPR Pair [*]

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According to Bell's theorem, the degree of correlation between spatially separated measurements on a quantum system is limited by certain inequalities if one assumes the condition of locality. Quantum mechanics predicts that this limit can be exceeded, making it nonlocal. We analyse the effect of an environment modelled by a fluctuating magnetic field on the quantum correlations in an EPR singlet as seen in the Bell inequality. We show that in an EPR setup, the system goes from the usual 'violation' of Bell inequality to a 'non-violation' for times larger than a characteristic time scale which is related to the parameters of the fluctuating field. We also look at these inequalities as a function of the spatial separation between the EPR pair.

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I. INTRODUCTION

The basic formalism of quantum theory which involves concepts of probability amplitudes and the superposition principle has been the source of many paradoxes and strange implications. A celebrated paradox that illustrates the counterintuitive conceptual framework of quantum mechanics is the 'E-P-R paradox' which is based on a gedankenexperiment proposed by Einstein, Podolsky and Rosen (E-P-R) in 1935 [1].

E-P-R showed that quantum theory is incomplete as the quantum description of the physical system does not contain all the 'elements of reality'. The so-called 'elements of reality' are those physical attributes of the system that can be measured without disturbing the system. E-P-R's argument also required that the principle of locality be obeyed in the sense that the disturbance caused by the act of measurement does not travel faster than light. It seems intuitive to expect all physical theories to be local and consistent with realism as stated above. But E-P-R's argument showed that quantum mechanics is inconsistent with the doctrines of realism and locality. A natural requirement at that time was, thus, to look for a theory that agreed with all the predictions of quantum mechanics and yet did not share its conflicts with realism and locality. One possibility was to reinterpret quantum mechanics in terms of a statistical account of an underlying hidden-variable theory which would conform to locality and realism. Many attempts were made to find such a complete 'hidden variable theory'. In 1965 Bell proved that nonlocal correlations among measured quantities must obey some general inequalities that are common to all local realistic theories [2]. Quantum mechanics, on the other hand, gives rise to correlations that violate Bell's inequalities. This provided definite experimental ways to test for any hidden-variable theories that may underlie quantum mechanics. A number of experiments have been carried out to measure such correlations and most of them have yielded results which are in excellent agreement with quantum theory and in disagreement with local realistic theories [3,4]

In the following we describe Bohm's version [5,6] of the E-P-R gedankenexperiment which was considered by Bell to prove his theorem. A bound state of two spin-1/2 particles is prepared in the quantum-mechanical singlet state $| \psi \rangle$ given by

$$| \psi \rangle = \frac{1}{\sqrt{2}} \left(|\uparrow_{1n}\rangle \otimes |\downarrow_{2n}\rangle - |\downarrow_{1n}\rangle \otimes |\uparrow_{2n}\rangle \right), \qquad (1)$$

where $|\uparrow_{in}\rangle$ and $|\downarrow_{in}\rangle, i = 1, 2$, describe the spin states in which particle i has spin 'up' and 'down', respectively, along the direction n. The bound state is now broken in such a way that the particles fly apart in opposite directions, but their total spin state remains a singlet. Two detectors, far apart from each other , are arranged to measure the spin component of particle 1 along a direction a and that of particle 2 along a direction b. The directions a and b are at an angle θ with respect to each other. According to the quantum formalism, the measured expectation value can be written as

$$E(a,b) \equiv \langle \psi \mid \sigma_1 . a \sigma_2 . b \mid \psi \rangle = -cos\theta. \qquad (2)$$

One can see that for $\theta = 0$, E(a ,b) = -1. This implies the existence of a perfect negative correlation between the spins of 1 and 2. Thus, one can predict with certainty the result of a measurement on 2 by previously obtaining the result of 1. This measurement on 1 does not disturb the state of the particle 2, which is very far from 1. Therefore, one argues that all components of the spin of 2 are 'elements of reality', which quantum mechanics cannot describe. Further, since the quantum mechanical state $| \psi \rangle$ does not determine the result of an individual measurement, there must exist a more complete specification of the state through a hidden-variable theory. By assuming a suitable condition of locality, Bell [2] showed that for any local, realistic, deterministic theory, the expectation value (2) must satisfy a simple inequality, one form of which is

$$| E(a,b) - E(a,c) | \leq 1 + E(b,c), \qquad (3)$$

where a , b and c are three directions along which the detectors measure the spin components of 1 and 2. One can easily see that for the condition $\theta(\text{a},b) = \theta(b,c) = \pi /3$ and $\theta(\text{a},c) = 2\pi /3$, quantum mechanics clearly violates this inequality, a result verified by many successful experiments [4, 6-7]. These results establish nonlocality as an inescapable 'fact of life' and cast doubts on the notion of 'elements of reality'. There are many exciting implications of quantum nonlocality like quantum cryptography and quantum computers. Particularly fascinating is the idea of 'teleportation' of a quantum state using E-P-R pairs [8].

However, we know from everyday experience that nonlocal correlation of the E-P-R kind are not seen in the macroscopic physical world. Classical systems are known to conform to realism and locality. How does this transition take place? How do nonlocal correlations disappear and how does classicality emerge? The transition from quantum to classical is marked by the loss of quantum coherence which reduces a pure state density matrix for the system to a classically interpretable diagonal mixture. In recent years this 'decoherence' [9,10] approach has provided a way to understand the emergence of classicality from underlying quantum dynamics. The approach recognizes that a macroscopic system is never isolated from its environment but is constantly interacting with it. In this approach one calculates the combined density matrix of the system and the environment but since one is interested only in monitoring the system's degrees of freedom, one traces over the environmental degrees of freedom. Then one finds that the reduced density matrix of the system is driven to a diagonal form in time.

In this paper we study the effect of an environment on an E-P-R singlet where the particles are acted upon by forces which push them apart in opposite directions. The effect of the environment here is two-fold. Firstly, the two particles are coupled to a bath of harmonic oscillators via a Caldeira-Leggett type of dissipative coupling through their positional degrees of freedom [9]. This aspect of the model has been discussed in detail elsewhere [10]. Secondly, the spin degrees of freedom for the two particles are coupled to an environment modelled by a fluctuating external magnetic field. We calculate the spin correlations for this system, and we show that the system goes from a 'violation' of the inequality at t=0 to a 'nonviolation' over a time scale related to the parameters of the fluctuating magnetic field. We also look at this transition as a function of the spatial separation between the E-P-R pair. In Section II we introduce the model and set up the equations that the reduced density matrix of the system obeys. In Section III we obtain the solutions for these equations and calculate the correlations involved in the Bell inequality. Finally, in Section IV we summarize the results of this paper.

II. THE MODEL

Our model consists of two particles whose initial state can be described by the following wave function

$$\psi = \phi(1)\phi(2) \otimes \frac{1}{\sqrt{2}}\bigg(|\uparrow_{1n}\rangle \otimes |\downarrow_{2n}\rangle - |\downarrow_{1n}\rangle \otimes |\uparrow_{2n}\rangle \bigg), \quad (4)$$

where $\phi(1)$, $\phi(2)$ represent the spatial parts of the wavefunction which are initially Gaussian wave packets with zero initial momenta and centered around x=0

$$\phi(x,0) = \frac{1}{(dr\pi)^{1/2}} exp\bigg(-x^2/2d^2 \bigg). \qquad (5)$$

The particles are coupled to an environment consisting of a collection of harmonic oscillators via their position coordinates. Such a model of the environment has been studied in great detail by many authors in the context of quantum dissipative systems and the quantum measurement problem [9-10]. In addition to this coupling, the spin degrees of freedom for the two particles are coupled to an environment modelled by a fluctuating magnetic field. The Hamiltonian for this model is:

$$\begin{aligned}
H = &\frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \epsilon X_1 - \epsilon X_2 \\
&+ \sum \frac{p_{1k}^2}{2m_{1k}} + \frac{m_{1k}\omega_{1k}^2}{2}\bigg(x_{1k} - \frac{c_{1k}X_1}{m_{1k}\omega_{1k}^2} \bigg)^2 \\
&+ \sum \frac{p_{2k}^2}{2m_{2k}} + \frac{m_{2k}\omega_{2k}^2}{2}\bigg(x_{2k} - \frac{c_{2k}X_2}{m_{2k}\omega_{2k}^2} \bigg)^2 \\
&- \frac{1}{2}g\hbar(B_1.\sigma_1 + B_2.\sigma_2), \qquad (6)
\end{aligned}$$

where P_1, X_1 and P_2, X_2 are the momentum and position coordinates of particles 1 and 2, respectively, ϵ is the strength of the applied force (the force being positive for 1 and negative for 2), p_{1k}, x_{1k} and p_{2k}, x_{2k} are the momentum and position coordinates of the kth harmonic oscillator of the baths which couple to 1 and 2, c_{1k} and c_{2k} are the respective coupling strengths and ω_{1k} and ω_{2k} are the frequencies of the oscillators. B_1 and B_2 are the external stochastic magnetic fields acting on particles 1 and 2, and g is the gyromagnetic ratio. Though in our model, we assume that the external fluctuating fields $B_1(t)$ and $B_2(t)$ are random in time, we envisage the physical origin of this randomness to arise due to spatial variation of the field in the region where the singlet is separating. Since the analytical treatment for the spatially varying field is very difficult, we model the spatial variation seen by the wavepackets of the particles to be a temporal variation. $B_1(t)$, $B_2(t)$ are taken to be Gaussian random processes, having the following correlations:

$$g^2 \langle B_{i\alpha}(t_1)B_{i\beta}(t_2) \rangle = \frac{1}{2\tau_{\alpha\alpha}}\delta_{\alpha\beta}\delta(t_1 - t_2), \qquad (7)$$

where $i = 1,2$ and β refer to the components of the vector B_i, $\tau_{\alpha\alpha}$ are the correlation times.. The reduced density matrix of the system, $\rho(x_1, s_1, y_1, s_1; x_2, s_2, y_2, s_2) =$

$\langle x_1, s_1; x_2, s_2 \mid \rho_R \mid y_1, s_1; y_2, s_2 \rangle$ can be obtained by using the Feynman-Vernon functional integral method in which one first writes the path-integral expression for the complete density matrix and then traces over the oscillator degrees of freedom. Under conditions of weak coupling to the oscillator variables, (i.e., the Markovian limit) one finds that ρ_R obeys an equation which can be written as

$$\frac{\partial \rho_R}{\partial t} = -i[L_{s1} + L_{s2}]\rho_R + ig/2[B_1.\sigma + B_2.\sigma, \rho_R]. \quad (8)$$

Here L_{s_1} and L_{s_2} are a form of quantum-Liouville operators which in spatial representation are given by

$$L_{sj} = \frac{-\hbar}{2m}\Big\{\frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial y_j^2}\Big\} - i\gamma(x_j - y_j)\Big\{\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j}\Big\}$$
$$- \frac{iD}{4\hbar^2}(x_j - y_j)^2 \pm \frac{i\epsilon}{\hbar} \quad (9)$$

where the upper sign is for particle 1 and the lower one for particle 2, and γ and D are related to the environment, playing the role analogous to friction and diffusion coefficients, respectively. Since the particles are noninteracting (apart from the correlation due to the initial condition) the density matrix equation is separable, i.e., $\rho_R = \rho_1 \otimes \rho_2$ with each of ρ_1 and ρ_2 satisfying the equation

$$\frac{\partial \rho_j}{\partial t} = -iL_{sj}\rho_j + \frac{ig}{2}[B_j.\sigma_j, \rho_j], j = 1, 2. \quad (10)$$

We can further effect the separation between the space and the spin degrees of freedom for each particle by recognizing that ρ_j is a 2x2 matrix in spin indices and can therefore be written as

$$\rho_j = \frac{1}{2}\Big(\rho_{sj}(x_j, y_j, t)\mathbf{1} + W_j(x_j, y_j, t).\sigma_j\Big), \quad (11)$$

where $\mathbf{1}$ denotes a 2x2 unit matrix. Substituting Eq. (11) in Eq. (10) one easily finds the dynamical equations governing ρ_{sj} and W_j :

$$\frac{\partial \rho_{sj}}{\partial t} = -iL_{sj}\rho_{sj}(x_j, y_j, t), \quad (12)$$

$$\frac{\partial W_j(x_j, y_j, t)}{\partial t} = -iL_{sj}W_j - gB_j x W_j. \quad (13)$$

From Eq.(11) it is clear that $W_j(x_j y_j; t) = \rho_{sj}(x_j, y_j, t)P_j$ and the polarization vector P obeys the equation

$$\frac{\partial P_j(t)}{\partial t} = gP_j x B_j. \quad (14)$$

Thus the complete solution for the problem can be written as

$$\rho_R(x_1, s_1, y_1, s_1; x_2, s_2, y_2, s_2, t) = \frac{1}{4}\rho_{s_1}(x_1, y_1, t)$$
$$\rho_{s_2}(x_2, y_2, t)$$
$$\times [\mathbf{1} + \sigma_1.P_1(t)]_{s_1 s_1'}$$
$$[\mathbf{1} + \sigma_2.P_2(t)]_{s_2 s_2'} \quad (15)$$

In the next Section we write down the solutions for equations (12) and (14) and calculate the correlations involved in the Bell inequality using Eq. (15).

III. THE DENSITY-MATRIX OF THE E-P-R PAIR

The solution for the spatial part of the density matrix, i.e., the solution of Eq. (12) have been worked out in detail elsewhere [10] . For the initial condition of Eq. (5), these solutions are

$$\rho_{sj}(R_j, r_j, t) = \sqrt{\frac{\pi}{M(\tau)}}exp\Big\{-\Big(\frac{1}{4d_j^2}e^{-2\tau}$$
$$+ \frac{D}{8\hbar^2}(1 - e^{-2\tau})\Big)r_j^2 \mp \frac{i\epsilon}{\hbar\gamma}(\epsilon^\tau)r_j$$
$$- \frac{1}{M(\tau)}\Big(R_j \pm \frac{\epsilon}{m\gamma^2}(1 - e^{-\tau} - \tau)$$
$$- \frac{i\hbar r_j}{2d_j^2 m\gamma}\epsilon^\tau(1 - e^{-\tau})$$
$$- \frac{iDr_j}{4m\gamma^2}\frac{}{\hbar}(1 - e^{-\tau})^2\Big)^2\Big\}, j = 1, 2 \quad (16)$$

where the upper signs are for $j = 1$ and the lower signs are for j=2, $\tau = \gamma t$, $R_j = (x_j + y_j)/2$, $r_j = x_j - y_j$, and

$$M(\tau) = d_j^2 + \frac{\hbar^2}{d_j^2 m^2 \gamma_2}(1 - e^{-\tau})^2$$
$$+ \frac{D}{2m^2\gamma^3}(2\tau - 3 + 4e^{-\tau} - e^{-2\tau}). \quad (17)$$

We first note that the off-diagonal part of ρ_{sj} corresponding to $r_j \neq 0$ vanishes rapidly in time. If we look at the diagonal parts of (16), i.e., the position distribution functions in the $t \to \infty$ limit, these solutions represent two Gaussian wave packets centered around $+\frac{\epsilon\tau}{m\gamma^2}$ and $-\frac{\epsilon\tau}{m\gamma^2}$ which are moving away from each other with time.

We now consider Eq. (14). Note that it is exactly the equation for a classical moment in an external magnetic field. Since the field $B(t)$ is stochastic, $P(t)$ will also be a stochastic variable. The quantity of interest corresponding to a real physical observation would, therefore, be the expectation value of $P(t)$, averaged over the ensemble of the random process $B(t)$. One has to, thus, find the solution to a stochastic Liouville equation , i.e., an equation for the probability distribution $f(P, t)$ for the stochastic variable $P(t)$ [11] . This problem has been studied extensively in the literature [11,12] . For the sake

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of completeness, we record the main results. The equation obeyed by the probability distribution function $f(P, t)$ averaged over the ensemble of the random process is

$$\frac{\partial f}{\partial t} = -Gf, \tag{18}$$

where

$$G = 1/2[L_x^2/2\tau_{xx} + L_y^2/2\tau_{yy} + L_z^2/2\tau_{zz}], \tag{19}$$

and the operator L is given by

$$L = -iPx\frac{\partial}{\partial P} \tag{20}$$

From Eq. (18) one can write down the equation for the average value of P as

$$\frac{\partial \langle P \rangle}{\partial t} = -\langle G^+ P \rangle, \tag{21}$$

which reduces to the well-known Bloch equations if we choose $\tau_{xx} = \tau_{yy} = \tau_1$ and $\tau_{zz} = \tau_0$ where the z-axix is chosen to be the axis of motion of the E-P-R pair. Eq (21) then yields

$$\frac{d\langle P_z \rangle}{dt} = -\frac{1}{2\tau_1}\langle P_z \rangle, \tag{22a}$$

$$\frac{d\langle P_\pm \rangle}{dt} = -\left(\frac{1}{4\tau_1} + \frac{1}{4\tau_0}\right)\langle P_\pm \rangle. \tag{22b}$$

where $P_\pm = P_x \pm iP_y$. The solutions to the Bloch equations (22) are

$$\langle P_z(t) \rangle = \langle P_z(0) \rangle exp(-t/2\tau_1), \tag{23}$$

$$\langle P_\pm(t) \rangle = \langle P_\pm(0) \rangle exp(-t/4\tau_1 - t/4\tau_0), \tag{24}$$

Let us now get back to the problem of the E-P-R singlet state (1) where the two spins are separately interacting with the external fluctuating magnetic fields. We note that the spin part of the density matrix corresponding to the initial singlet state can be written in the following manner:

$$\rho(0) = |\psi\rangle\langle\psi|$$
$$= \frac{1}{2}\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}_1\begin{pmatrix}0&0\\0&1\end{pmatrix}_2 + \begin{pmatrix}0&0\\0&1\end{pmatrix}_1\begin{pmatrix}1&0\\0&0\end{pmatrix}_2\right.$$
$$\left.- \begin{pmatrix}0&1\\0&0\end{pmatrix}_1\begin{pmatrix}0&0\\1&0\end{pmatrix}_2 - \begin{pmatrix}0&0\\1&0\end{pmatrix}_1\begin{pmatrix}0&1\\0&0\end{pmatrix}_2\right\}, \tag{25}$$

which can be expressed in terms of the Pauli spin operators of particles 1 and 2 as

$$\rho(0) = \frac{1}{4}\left(\mathbf{1} \otimes \mathbf{1} - \sigma_1^x\sigma_2^x - \sigma_1^y\sigma_2^y - \sigma_1^z\sigma_2^z\right)$$
$$= \frac{1}{4}\left(\mathbf{1} \otimes \mathbf{1} - \sigma_1.\sigma_2\right). \tag{26}$$

This is a convenient form for the initial condition, as we note that for calculating the correlation between particles 1 and 2, the spin part of our solution in Eq. (15) can be written as

$$\rho_{spin} = \frac{1}{4}\left(\mathbf{1} \otimes \mathbf{1} + \sigma_1.P_1(t)\sigma_2.P_2(t)\right). \tag{27}$$

Thus the initial conditions can be regarded as the superposition of three initial condition corresponding to $P_1 = x = -P_2; P_1 = y = -P_2$ and $P_1 = z = -P_2$. Thus, solving for the time evolution of ρ involves solving for the various components of $P($ as in (23), (24)) for each of the spins with these initial conditions separately and substituting into Eq.(26). One can easily see that the time dependent solution is:

$$\rho(t) = \frac{1}{4}\left(\mathbf{1} \otimes \mathbf{1} - (\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y)exp(-t/4\tau_0^{(1)}\right.$$
$$-t/4\tau_1^{(1)} - t/4\tau_0^{(2)} - t/4\tau_1^{(2)})$$
$$\left.-\sigma_1^z\sigma_2^z \exp(-t/2\tau_0^{(1)} - t/2\tau_0^{(2)})\right), \tag{28}$$

where $\tau_0^{(1)}, \tau_0^{(2)}, \tau_1^{(1)}, \tau_1^{(2)}$ are the characteristic time scales for the particles 1 and 2. We can now calculate the expectation values required in the Bell inequalities.

$$\langle \sigma_1.n_1\sigma_2.n_2 \rangle = -(n_{1x}n_{2x} + n_{1y}n_{2y})\exp(-\frac{t}{4}(1/\tau_0^{(1)}$$
$$+1/\tau_1^{(1)} + 1/\tau_0^{(2)})) + 1/\tau_1^{(2)}n_{1z}n_{2z}$$
$$\exp(-t/2\tau_0^{(1)} - t/2\tau_0^{(2)}). \tag{29}$$

For simplicity, if we assume that all the time scales are the same, i.e., $\tau_0^{(1)} = \tau_0^{(2)} = \tau_1^{(1)} = \tau_1^{(2)} = \tau_s$, then one can easily see that the expectation value (29) becomes:

$$\langle \sigma_1.n_1\sigma_2.n_2 \rangle = Trace(\rho(t)\sigma_1.n_1\sigma_2.n_2)$$
$$= -cos\theta exp(-t/\tau_s), \tag{30}$$

transforming the Bell inequality (3) to

$$|E(a,b) - E(a,c)| exp(-t/\tau_s) \leq 1 + E(b,c)exp(-t/\tau_s). \tag{31}$$

This shows that as soon as the left-hand side becomes smaller than unity, the inequalities are satisfied. For the choice of angles $\theta(a,b) = \theta(b,c) = \pi/3$ and $\theta(a,c) = 2\pi/3$, the system goes from a violation of the inequality to a nonviolation at time $t = -\tau_s ln(2/3)$. Using the solution of the spatial part, i.e., Eq (16), this means that the quantum correlations are lost when the separation between the detectors is of the order of $\frac{2\epsilon}{m\gamma}\tau_s ln(3/2)$. As mentioned earlier, the origin of the randomness of the magnetic field is being attributed to the spatial randomness of the field as seen by the traversing wave-packets of each particle. It is clear that for E-P-R correlations to persist for longer and longer times, one requires isotropy

or absence of field fluctuations in larger and larger portions of space. Indeed, it is important to keep in mind that this instantaneous quantum propagation of information from one particle to another is contingent upon the requirement that the intervening space is free from all field fluctuations − a requirement that has to be locally ensured. Another odd feature worth mentioning is that the decoherence in the spatial part and the spin part are completely unrelated. Thus, even when the probability distributions of position of the two particles have become classical and well-separated, the spin wave-function may continue to have nonlocal correlations - a situation one would not expect classically.

IV. SUMMARY

To summarize, in this paper we have studied the effect of the environment on an E-P-R setup. To simulate an E-P-R setup, we assume that the two particles of the E-P-R singlet are acted upon by forces which move them in opposite directions. We find that environmental influence causes decoherence in the quantum correlations of the initial singlet state leading to a nonviolation of Bell's inequalities, thus restoring the classical character of the correlations. We have also tried to correlate this loss of coherence to the spatial separation between the two particles which are additionally coupled to an environment through the Caldeira-Leggett type of dissipative coupling.

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